The boundary of the first order chiral phase transition in the $m_\pi - m_K$–plane with a linear sigma model

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Tree-level and complete one-loop parametrisation of the linear sigma model (LσM) is performed and the phase boundary between first order and crossover transition regions of the $m_\pi - m_K$–plane is determined using the optimised perturbation theory (OPT) as a resummation tool of perturbative series. Away from the physical point the parameters of the model were determined by making use of chiral perturbation theory (ChPT). The location of the phase boundary for $m_\pi = m_K$ and of the tricritical point (TCP) on the $m_\pi = 0$ were estimated.

1. Introduction

Lattice studies show that at the physical point of the $m_{u,d} - m_s$–plane the transition as a function of the temperature is of crossover-type [1]. In the $m_{u,d} = m_s = 0$ limit effective models predict a first order phase transition and for $m_s = \infty$ and $m_{u,d} = 0$ a second order phase transition [2]. Theoretically it is interesting to know where the boundary of the first order region lies and where the TCP is located on the $m_u = 0$ axis.

Several lattice studies with degenerate quarks ($m_{u,d} = m_s$) show that the critical pion mass on the boundary decreases substantially, from $m_\pi^c \approx 290$ MeV to $m_\pi^c = 67(18)$ MeV, as finer lattices and improved actions are used (see *e.g.* [3, 4]). A low pion mass allows for studies in effective models based on the chiral symmetry because it is expected that they work better closer to the chiral limit. Existing works performed in these models give also a low value for the critical pion mass [5, 6, 7]. In these works only the Goldstone masses were tuned, the other parameters of the Lagrangian were kept at their values determined at the physical point. We think that in this low mass regime effective model studies might complement lattice investigations if coupling parameters can be determined accurately. Hence we let the coupling constants to vary with the pion and kaon masses when moving away from the physical point and determined their values using the low energy results of the ChPT, namely the dependence of f_π, f_K, m_η and $m_{\eta'}$ on m_π and m_K.

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2. The parametrisation of the linear sigma model

The Lagrangian is written in terms of a 3×3 matrix $M=\lambda_a(\sigma_a+i\pi_a)/\sqrt{2}$ ($\lambda_a: a=1\dots 8$ are the Gell-Mann matrices and $\lambda_0:=\sqrt{2/3}\mathbf{1}$) containing the nonet of scalar (σ_a) and pseudoscalar (π_a) particles:

$$\begin{aligned} L(M) &= \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M^\dagger\partial^\mu M+\mu^2 M^\dagger M\right)-f_1\left(\mathrm{Tr}\left(M^\dagger M\right)\right)^2-f_2\mathrm{Tr}\left(M^\dagger M\right)^2 \\ &\quad -g\left(\det(M)+\det(M^\dagger)\right)+\epsilon_0\sigma_0+\epsilon_8\sigma_8. \end{aligned} \tag{1}$$

The determinant term breaks the axial $U(1)$ symmetry and the external fields are introduced in order to give masses to the pseudo-Goldstone bosons. From the original 0-8 basis one can switch with an orthogonal transformation to a basis in which in the broken symmetry phase the two non-vanishing expectation values are the non-strange ($x=(\sqrt{2}\langle\sigma_0\rangle+\langle\sigma_8\rangle)/\sqrt{3}$) and the strange ($y=(\langle\sigma_0\rangle-\sqrt{2}\langle\sigma_8\rangle)/\sqrt{3}$) condensates.

2.1. Tree-level parametrisation

The tree-level parametrisation of the model can be performed at zero temperature by solving a set of coupled linear equations [8, 9]. Only the trace of the mass matrix is used in the mixing $\eta-\eta'$ sector. Unfortunately the values the quartic coupling f_1 and the mass parameter μ^2 can be obtained using only the scalar sector which compared to the pseudoscalar sector is experimentally less known. Besides, one knows nothing about the m_π and m_K-dependence of the scalar masses and as a consequence some assumptions are needed. For example in [9] we required the Gell-Mann-Okubo mass formula to be satisfied in the scalar sector on the entire $m_\pi-m_K$–plane with the same accuracy as at the physical point. Performing a consistency check by comparing m_η and $m_{\eta'}$ calculated in the LσM with their values given by the ChPT remarkable agreement was obtained up to a limiting value of $m_K\approx 800$ MeV, even away from the physical point.

2.2. One-loop parametrisation

The parametrisation at the one-loop level of the perturbation theory is more complicated and there are many possible ways of selecting the set of non-linear equations. One such set of equations is presented below (see [10] for details).

Because in the broken symmetry phase the tree-level mass can be negative a resummation is needed. We performed this using the optimised perturbation theory method of Ref. [11] where a mass term is added to and subtracted from the Lagrangian, the difference between the original and the new mass is treated perturbatively (first at one-loop level). The new mass is determined from the principle of minimal sensitivity which requires that the one-loop pole mass for the pion (M_π) stays equal to its tree-level value (m_π). This can be translated into a self-consistent equation for the pion mass

$$m_\pi^2=-\mu^2+(4f_1+2f_2)x^2+4f_1y^2+2gy+\mathrm{Re}\Sigma_\pi(p^2=m_\pi^2). \tag{2}$$

Two more equations are the one-loop pole mass of the kaon and η (smaller mass eigenvalue in the mixing sector):

$$M_K^2=-\mu^2+2(2f_1+f_2)(x^2+y^2)+2f_2y^2-\sqrt{2}x(2f_2y-g)+\mathrm{Re}\Sigma_K(p^2=M_K^2), \tag{3}$$

$$\mathrm{Det}\begin{pmatrix} p^2-m^2_{\eta_{xx}}-\Sigma_{\eta_{xx}}(p^2) & -m^2_{\eta_{xy}}-\Sigma_{\eta_{xy}}(p^2) \\ -m^2_{\eta_{xy}}-\Sigma_{\eta_{xy}}(p^2) & p^2-m^2_{\eta_{yy}}-\Sigma_{\eta_{yy}}(p^2)\end{pmatrix}\Bigg|_{p^2=M_\eta^2}=0. \tag{4}$$

We require additionally the equality of the one-loop an tree-level kaon masses:

$$M_K^2 = m_K^2 = m_\pi^2 - 2gy + 4f_2y^2 - \sqrt{2}x(2f_2y - g), \tag{5}$$

and also make use of the PCAC relations:

$$f_\pi = Z_\pi^{-\frac{1}{2}} M_\pi^{-2}(-iD_\pi^{-1}(p=0))x, \qquad f_K = Z_K^{-\frac{1}{2}} M_K^{-2}(-iD_K^{-1}(p=0))(x+\sqrt{2}y)/2. \tag{6}$$

Away from the physical point we use for the determination of m_η, f_π and f_K the formulae of the SU(3) ChPT in the large-N_c limit. Because in the one-loop parametrisation the renormalisation scale (l) appears as a new parameter we checked how the other parameters and masses depends on it. The wave function renormalisation constants have a plateau and the variation of the one-loop masses is the mildest in the range $l \in (1000, 1400)$ MeV, which was selected to be used for the determination of the phase boundary.

3. Thermodynamics of the LσM

The order of the phase transition is determined by solving simultaneously the self-consistent gap equation for the pion mass

$$m_\pi^2 = -\mu^2 + (4f_1 + 2f_2)x^2 + 4f_1y^2 + 2gy + \Sigma_\pi(p^2 = m_\pi^2, m_i(m_\pi), l), \tag{7}$$

where $\Sigma_\pi(p, m_i(m_\pi), l)$ is the pion self-energy, and the two equations of state

$$\begin{aligned} \epsilon_x &= -\mu^2 x + 2gxy + 4f_1xy^2 + (4f_1 + 2f_2)x^3 + \textstyle\sum_i J_i t_i^x(x,y)\mathrm{I}_{\mathrm{tp}}[m_i(m_\pi), T], \\ \epsilon_y &= -\mu^2 y + 2gx^2 + 4f_1x^2y + (4f_1 + 4f_2)y^3 + \textstyle\sum_i J_i t_i^y(x,y)\mathrm{I}_{\mathrm{tp}}[m_i(m_\pi), T]\,. \end{aligned} \tag{8}$$

The isospin multiplicity factor J_i and the coefficients t_i^x and t_i^y are given in Ref. [9].

The difference between the tree-level and one-loop parametrisation is that in the first case one are enforced to throw out entirely the vacuum fluctuations and use only the temperature dependent part of the integrals, while in the second case the renormalised vacuum fluctuations are taken into account. The difference between the two methods of solving the model was checked quantitatively in [10].

4. The phase-boundary in the $m_\pi - m_K$–plane

In the case of the tree-level parametrisation the result on the phase boundary is not too conclusive because of the dependence of the phase boundary on the assumptions made for the scalar sector in the process of parametrisation. We have not seen the trace of the tricritical point. We estimated that the phase boundary crosses the diagonal of the $m_\pi - m_K$-plane for critical pion mass in the range $m_\pi^c = 40 \pm 20$ MeV.

As for the one-loop parametrisation, along the diagonal of the pion-kaon mass plane we estimate $m_\pi^c = 110 \pm 20$ MeV (see Fig. 1). At high values of m_K the phase boundary is not renormalisation scale dependent. We can clearly see the scaling region of the trictitical point as well. Unfortunately we were not able to locate it directly because the model parametrisation breaks down just before arriving at the TCP along the $m_\pi = 0$ line, but based on the scaling near TCP we estimate it to be in the interval $m_K^{TCP} \in \{1700, 1850\}$ MeV. We can directly compare our result on the location of the TCP with the recent

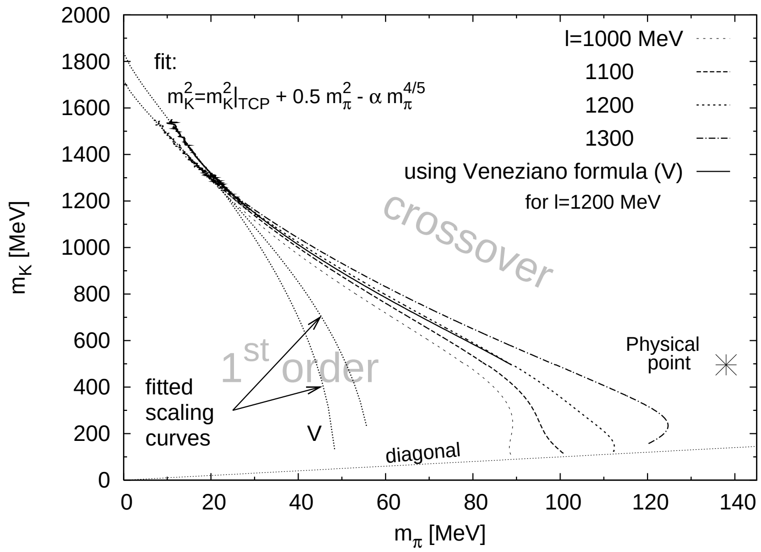

Figure 1. Phase boundary of the LσM obtained using a one-loop parametrisation at $T = 0$ and the formulae of large N_c ChPT for continuation on the $m_\pi - m_K$–plane.

lattice result of Ref. [12] by showing the phase boundary in the quark mass-plane. In Ref. [12] the physical point is more close to the boundary than in our case and the value of the mass of the strange quark at the TCP is $m_s^{\rm TCP} = 3m_s$ ($m_s^{\rm phys}$ is the value of the strange quark mass at the physical point). In our case $m_s^{\rm TCP} = 13 - 15 \times m_s^{\rm phys}$. In Ref. [12] the location of the TCP was estimated also by fitting with the corresponding scaling equation ($m_{u,d} \approx (m_s^{\rm TCP} - m_s)^{5/2}$), but only points with $m_s \leq m_s^{\rm phys}$. The investigation performed in the LσM shows that the scaling region sets in very close to the $m_u = 0$ ($m_\pi = 0$) axis. It would be interesting to know how the lattice estimate would change if points more closer to the $m_u = 0$ axis were available.

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